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A multifractal decomposition according to rate of returns

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For one-dimensional simple symmetric random walk, the Hausdorff and packing dimensions of sets of sample paths with prescribed rate of returns to the origin are determined. This gives a multifractal decomposition of the underlying sample space.

1 Introduction

Let $\{\pm 1\}^{\mathbb{N}}$ be the space of infinite sequences of ±1's, and put

$$S_n(x) := \sum_{k=1}^{n} x_k \quad \text{and} \quad R_n(x) := \sharp\{k \le n : S_k(x) = 0\}$$

for $x = (x_k)_{k=1}^{\infty} \in \{\pm 1\}^{\mathbb{N}}$ and $n \ge 1$. Here and throughout the symbol "$\sharp$" stands for "the cardinality of".

Suppose that the space $\{\pm 1\}^{\mathbb{N}}$ is endowed with the Bernoulli measure $\left(\frac{1}{2}, \frac{1}{2}\right)^{\mathbb{N}}$, then $S_n(.)$ may be thought of being the position after n steps for the simple symmetric random walk on the integer lattice $\mathbb{Z}$, and $R_n(.)$ being the total number of returns to the origin during the first n steps. Various aspects of the theory of this walk are accounted in Feller's book [5], including the well-known assertion of Pólya: with probability one,

$$\lim_{n \to \infty} R_n(.) = \infty.$$

In words, the random walk will return to the origin infinitely often.

It is natural to ask: What can be said about the rate of growth of $R_n(.)$ as $n \to \infty$? Since $S_n(.) = 0$ only if n is even, it is immediately seen that

$$0 \le R_{2n-1}(.) \le R_{2n}(.) = R_{2n+1}(.) \le n, \quad \text{for all} \quad n \ge 1.$$

Moreover, in Kesten [7], it was shown that: with probability one,

$$\limsup_{n \to \infty} \frac{R_n(.)}{\sqrt{2n \log\log n}} = 1 \quad \text{and} \quad \liminf_{n \to \infty} \frac{\sqrt{\log\log n}\, R_n(.)}{\sqrt{n}} = c,$$

where c is a positive constant. Apart from this, let us mention that the exact and the limit distribution of the $R_n(.)$ are also known. These facts can be found, for example, in the book of Révész [9, pp. 93–114].

To summarize, what is known is that, almost surely, the rate of growth of $R_n(.)$ is nearly $\sqrt{n}$. Then what can be said about the exceptional points? By "exceptional points" here we mean those sample points with rates

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of return different from $\sqrt{n}$. This leads us to consider three types of sets classified according to their return behaviors:

(i) sets of points for which the returns occur finitely often:

$$D_m := \left\{ x \in \{\pm 1\}^{\mathbb{N}} : \lim_{n\to\infty} R_n(x) = m \right\},$$

where $m \in \mathbb{N} \cup \{0\}$. In particular, the set D_0 consists of points of no returning.

(ii) sets of *slow* points:

$$E_\varphi := \left\{ x \in \{\pm 1\}^{\mathbb{N}} : \lim_{n\to\infty} \frac{R_n(x)}{\varphi(n)} = 1 \right\},$$

where the function $\varphi : [a, \infty) \to (0, \infty)$ $(a \geq 1)$ satisfies

$$\varphi'(t) > 0, \quad \lim_{t\to\infty} \varphi'(t) = 0 \tag{1.1}$$

and

$$\lim_{t\to\infty} f(t) = +\infty, \quad \lim_{t\to\infty} \frac{\log f(t+1)}{f(t)} = 0, \tag{1.2}$$

where $f := \varphi^{-1}$ is the inverse of φ. Typical examples of such functions are:

$$\alpha t^\beta, \quad \log t, \quad \log\log t, \quad \log\log\log t, \quad \ldots,$$

where $\alpha > 0$ and $0 < \beta < 1$;

(iii) sets of *quick* points:

$$F_\gamma := \left\{ x \in \{\pm 1\}^{\mathbb{N}} : \lim_{n\to\infty} \frac{R_n(x)}{n} = \gamma \right\},$$

where $0 \leq \gamma \leq \frac{1}{2}$.

With the standard metric (see (2.1) below), the sample space $\{\pm 1\}^{\mathbb{N}}$ is a compact metric space. Therefore, we may speak of Hausdorff, packing and upper box-counting dimension of arbitrary subsets. We shall denote them by $\dim_H$, $\dim_P$ and $\overline{\dim}_B$, respectively. For a nice account of the theory of fractal geometry, we refer to Falconer's book [3]. Not surprisingly, in accordance with the diverse return behavior of the simple random walk, the space of sample paths exhibits a rather rich multi-fractal structure. To show this, we shall determine the fractal dimensions of the sets just defined.

Let us introduce a pair of functions which will be useful in the sequel:

$$H(t) := -t\log t - (1-t)\log(1-t), \quad 0 \leq t \leq 1, \tag{1.3}$$

and

$$h(t) := t\log 2 + (1-t)H\left(\frac{1}{2-2t}\right), \quad 0 \leq t \leq \frac{1}{2}, \tag{1.4}$$

where $0\log 0 = 0$ by convention.

Our main result is the following

Theorem 1.1 *With the above notation, one has*

(i) *for any* $m \in \mathbb{N} \cup \{0\}$

$$\dim_H(D_m) = 1;$$

(ii) *for any* φ *satisfying* (1.1) *and* (1.2)

$$\dim_H(E_\varphi) = 1;$$

(iii) *for any* $0 \leq \gamma \leq \frac{1}{2}$

$$\dim_H(F_\gamma) = \dim_P(F_\gamma) = \frac{h(\gamma)}{\log 2}.$$

Some remarks are in order now.

Remark 1.2 Equipped with the metric defined in (2.1), the whole space $\{\pm 1\}^{\mathbb{N}}$ has Hausdorff dimension one. Actually, Theorem 1.1 may be regarded as a multi-fractal decomposition of the space $\{\pm 1\}^{\mathbb{N}}$.

• From (i) and (ii), it is seen that each set of the first two families in Theorem 1.1 has full dimension. Representative examples of such sets are

$$\begin{aligned}
&\bigl\{x \in \{\pm 1\}^{\mathbb{N}} : R_n(x) = 0, \text{ for all } n \geq 1\bigr\}, \\
&\left\{x \in \{\pm 1\}^{\mathbb{N}} : \lim_{n\to\infty} \frac{R_n(x)}{\log\log\log n} = 1\right\}, \\
&\left\{x \in \{\pm 1\}^{\mathbb{N}} : \lim_{n\to\infty} \frac{R_n(x)}{n^{2/3}} = 1\right\}.
\end{aligned}$$

These sets are null sets in the sense of probability, but all of them have full dimension.

• By elementary calculus, one may show that the function $h(\gamma)$ is strictly decreasing in $\bigl[0, \frac{1}{2}\bigr]$. The set $F_{1/2}$ has maximal rate of returns $\bigl($which is $\frac{1}{2}\bigr)$ and minimal dimension $\bigl($which is also $\frac{1}{2}\bigr)$.

• By (iii), for any $0 \leq \gamma \leq \frac{1}{2}$, write

$$F_\gamma = \left\{x \in \{\pm 1\}^{\mathbb{N}} : \lim_{n\to\infty} \frac{R_n(x)}{n} = \gamma\right\},$$

then

$$\dim_H(F_\gamma) = \dim_P(F_\gamma) = \frac{h(\gamma)}{\log 2}.$$

This reminds us of a theorem of Besicovitch [1], saying that: *For any* $0 < p < 1$*, the set*

$$\left\{x \in \{\pm 1\}^{\infty} : \lim_{n\to\infty} \frac{S_n(x)}{n} = p\right\}$$

has Hausdorff dimension $H\bigl(\frac{HP}{2}\bigr)\big/\log 2$. For some far-reaching generalizations of the theorem of Besicovitch, we refer to the recent works in [4] and [8] and the references therein.

The remaining part consists of two sections which are devoted respectively to preliminary lemmas and proofs of Theorem 1.1.

2 Preliminary lemmas

A usual metric on the sample space $\{\pm 1\}^{\mathbb{N}}$ is given by

$$d(x, y) = 2^{-\min\{k : x_{k+1} \neq y_{k+1}\}} \tag{2.1}$$

for $x, y \in \{\pm 1\}^{\mathbb{N}}$. With this metric, the balls are just cylinders of the following form

$$[x_1, \ldots, x_n] := \bigl\{y \in \{\pm 1\}^{\mathbb{N}} : y_1 = x_1, \ldots, y_n = x_n\bigr\}$$

which is of diameter 2^{-n}. Conventionally, we call $\{\pm 1\}^{\mathbb{N}}$ the symbolic space over the symbols ± 1; we call $\omega := (\omega_1, \ldots, \omega_n)$ a finite word of length $|\omega| = n$, and denote by $[\omega]$ the corresponding cylinder; for two finite words ω^1 and ω^2, we denote their concatenation by $\omega^1 \star \omega^2$ which is a word of length $\bigl|\omega^1\bigr| + \bigl|\omega^2\bigr|$; if $\mathcal{W}$ is a collection of finite words, then $[\mathcal{W}] := \bigcup_{\omega \in \mathcal{W}} [\omega]$ stands for the union of the corresponding cylinders.

2.1 Some combinatorial and asymptotic formulas

For any integers m and n with $0 \leq m \leq \frac{n}{2}$, define

$$\mathcal{C}_n(m) := \{(x_1, \ldots, x_n) : R_n(x) = m\} \quad \text{and} \quad [\mathcal{C}_n(m)] := \bigcup_{\omega \in \mathcal{C}_n(m)} [\omega]. \tag{2.2}$$

It is therefore seen that, the set $[\mathcal{C}_n(m)]$ is a union of n-cylinders, consisting of points for which the total number of returns to the origin during the first n steps is exactly m. By requiring additionally that the last return occurs on the $2n$-th step, we introduce the following notation:

$$\widetilde{\mathcal{C}}_{2n}(m) := \{(x_1, \dots, x_{2n}) : R_{2n}(x) = m, S_{2n}(x) = 0\}. \tag{2.3}$$

For the cardinalities of $\mathcal{C}_n(m)$ and $\widetilde{\mathcal{C}}_n(m)$, one has:

Lemma 2.1 *For any* $0 \le m \le n$, *one has*

$$\sharp\mathcal{C}_{2n}(m) = 2^m \binom{2n-m}{n}, \quad \sharp\mathcal{C}_{2n+1}(m) = 2^{m+1} \binom{2n-m}{n}$$

and

$$\sharp\widetilde{\mathcal{C}}_{2n}(m) = \frac{m.2^m}{2n-m} \binom{2n-m}{n}.$$

Proof. See [5, Problem 10 at page 96 and Theorem 4 at page 90]. □

In the sequel, we shall use the O-notation and o-notation in the usual sense, and the following consequence of the Stirling's formula

$$\log \binom{n}{m} = nH\Big(\frac{m}{n}\Big) + O(\log n), \tag{2.4}$$

where $H(.)$ is the entropy function in (1.3), but the constant implied by the O-notation may depend on the m.

The following technical lemma is important in the present paper.

Lemma 2.2 *Retaining the notations above, one has:*

(a) *If* $0 \le \beta \le \frac{1}{2}$, *and* $\beta n \le m \le \frac{1}{2} n$, *then*

$$\log \sharp\mathcal{C}_n(m) \le nh(\beta) + o(n);$$

(b) *If* $0 \le \gamma \le \frac{1}{2}$, *and* $m = 2\gamma n + O(1)$, *then*

$$\log \sharp\widetilde{\mathcal{C}}_{2n}(m) = 2nh(\gamma) + O(\log n).$$

Proof. (a) Write $m = \alpha n$, then $\beta \le \alpha \le \frac{1}{2}$. By Lemma 2.1 and (2.4), for $n = 2k$, one has

$$\begin{aligned}
\log \sharp\mathcal{C}_{2k}(m) &= m \log 2 + \log \binom{2k-m}{k} \\
&= m \log 2 + (2k-m) H\Big(\frac{k}{2k-m}\Big) + O(\log(2k-m)) \\
&= nh(\alpha) + O(\log n) \\
&\le nh(\beta) + o(n),
\end{aligned}$$

since $h(t)$ (defined in (1.4)) is decreasing in $\big[0, \frac{1}{2}\big]$. The case that $n = 2k+1$ can be treated in like manner.

(b) Using Lemma 2.1 and (2.4) again. □

2.2 Lower bounds for dimensions of certain subsets

To find a lower bound for the Hausdorff dimension of a complicated set, a practical way is to extract from it a subset which is simple enough in structure to be able to estimate its dimension from below. In the proof of Theorem 1.1, we shall construct two different classes of subsets. The first of these is the class of Moran sets which are generalization of the middle-third Cantor set: For each $n \ge 1$, let $\mathcal{W}_n$ be a collection of finite words (consisting of $+1$'s and -1's) with equal length $\ell_n \ge 1$.

Definition 2.3 A *Moran set adapted to* $\{\mathcal{W}_n\}_{n=1}^{\infty}$ is defined as

$$A_{\mathcal{W}} := \left\{ \left(\omega^1 \star \omega^2 \star \omega^3 \star \ldots\right) \in \{\pm 1\}^{\mathbb{N}} : \omega^n \in \mathcal{W}_n, \text{ for all } n \geq 1 \right\}.$$

In words, the set $A_{\mathcal{W}}$ consists of infinite words generated by successive concatenations of the words from $\{\mathcal{W}_n\}_{n=1}^{\infty}$. The following lower bound for the dimension of $A_{\mathcal{W}}$ can be found in [6, Theorem 2.1]:

$$\dim_H(A_{\mathcal{W}}) \geq \liminf_{n\to\infty} \frac{\sum_{k=1}^{n} \log \sharp \mathcal{W}_k}{\log 2 \sum_{k=1}^{n+1} \ell_k - \log \sharp \mathcal{W}_{n+1}}. \tag{2.5}$$

In the proof of Theorem 1.1, we shall employ the following construction: for each $n \geq 1$, take

$$\mathcal{W}_n = \widetilde{\mathcal{C}}_{\ell_n}(m_n) = \{(x_1, \ldots, x_{\ell_n}) : R_{\ell_n}(x) = m_n, S_{\ell_n}(x) = 0\}$$

with ℓ_n even and $1 \leq m_n \leq \frac{1}{2}\ell_n$ (see (2.3)). Denoting the resulting Moran set by $A(\{\ell_n, m_n\}_{n=1}^{\infty})$, we have

Lemma 2.4 (a) *If* $x \in A(\{\ell_n, m_n\}_{n=1}^{\infty})$ *and* $\sum_{k=1}^{n} \ell_k \leq N < \sum_{k=1}^{n+1} \ell_k$, *then*

$$\sum_{k=1}^{n} m_k \leq R_N(x) \leq \sum_{k=1}^{n+1} m_k;$$

(b) *If* $\lim_{n\to\infty} \frac{\ell_{n+1}}{\ell_1 + \cdots + \ell_n} = 0$, *then*

$$\dim_H(A(\{\ell_n, m_n\}_{n=1}^{\infty})) \geq \liminf_{n\to\infty} \frac{\sum_{k=1}^{n} \log \sharp \widetilde{\mathcal{C}}_{\ell_k}(m_k)}{\log 2 \sum_{k=1}^{n} \ell_k};$$

(c) *If* $m_n = O(1)$ *and* $\lim_{n\to\infty} \frac{\log \ell_{n+1}}{\ell_1 + \cdots + \ell_n} = 0$, *then*

$$\dim_H(A(\{\ell_n, m_n\}_{n=1}^{\infty})) = 1.$$

Proof. (a) By definition of $A(\{\ell_n, m_n\}_{n=1}^{\infty})$, one has

$$R_{\ell_1 + \cdots + \ell_n}(x) = \sum_{k=1}^{n} m_k, \quad \text{for all} \quad n \geq 1$$

implying the desired result.

(b) It is clear from (2.5).

(c) Put $\gamma = 0$ in (b) of Lemma 2.2. Since $h(0) = \log 2$, one has

$$\log \sharp \widetilde{\mathcal{C}}_{\ell_k}(m_k) = \ell_k \log 2 + O(\log \ell_k), \quad \text{for all} \quad k \geq 1.$$

It follows that

$$\sum_{k=1}^{n} \log \sharp \widetilde{\mathcal{C}}_{\ell_k}(m_k) = \log 2 \sum_{k=1}^{n} \ell_k + O\left(\sum_{k=1}^{n} \log k \right)$$

and

$$\log 2 \sum_{k=1}^{n+1} \ell_k - \log \sharp \widetilde{\mathcal{C}}_{\ell_{n+1}}(m_{n+1}) = \log 2 \sum_{k=1}^{n} \ell_k + O(\log \ell_{n+1}).$$

Putting these and $\lim_{n\to\infty} \frac{\log \ell_{n+1}}{\ell_1 + \cdots + \ell_n} = 0$ into (2.5), the desired result follows. □

The second class of subsets were studied by Fan and Schmeling [4] in the context of ergodic averages. Among other things, they proved the following:

Lemma 2.5 *Let $M \geq 1$ be an integer, and*

$$B_M := \left\{x \in \{\pm 1\}^{\mathbb{N}} : |S_n(x)| \leq M, \textit{ for all } n \geq 1\right\}. \tag{2.6}$$

Then one has that

$$\lim_{M\to\infty} \dim_H(B_M) = 1.$$

Actually, we may even prove that

$$\dim_H(B_M) = 1 + \log\cos\frac{\pi}{2M+1}.$$

3 Proofs of the theorem

An important ingredient in the proof is the construction of appropriate subsets with prescribed rate of returns and dimension.

3.1 Proof of (i)

For any $M \in \mathbb{N}$, put

$$B_M = \left\{x \in \{\pm 1\}^{\mathbb{N}} : |S_n(x)| \leq M, \text{ for all } \quad n \geq 1\right\}.$$

We are going to construct a subset of the set

$$D_m = \left\{x \in \{\pm 1\}^{\mathbb{N}} : \lim_{n\to\infty} R_n(x) = m\right\}.$$

Define a mapping $\tau : \{\pm 1\}^{\mathbb{N}} \to \{\pm 1\}^{\mathbb{N}}$ by

$$\tau(x_1, x_2, \dots) = (\underbrace{1, -1, 1-1, \dots, 1, -1}_{2m}, \underbrace{1, 1, \dots, 1}_{M+1}, x_1, x_2, \dots).$$

For any $x = (x_1, x_2, \dots) \in \{\pm 1\}^{\mathbb{N}}$, we have

$$R_{2m}(\tau(x)) = m \quad \text{and} \quad S_{2m+M+1}(\tau(x)) = M + 1.$$

It follows that, for any $x \in B_M$

$$S_n(\tau(x)) \geq 1 \quad \text{and} \quad R_n(\tau(x)) = m, \quad \text{for all} \quad n > 2m.$$

It is therefore seen that $\tau(B_M) \subset D_m$ for all $M \geq 1$. By the definition of the metric in (2.1), we see that

$$d(\tau(x), \tau(y)) = 2^{-(2m+M+1)} d(x, y), \quad \text{for all} \quad x, y \in \{\pm 1\}^{\mathbb{N}}.$$

This implies that the mapping τ preserves the Hausdorff dimension, hence

$$\dim_H(D_m) \geq \dim_H(\tau(B_M)) = \dim_H(B_M).$$

This is true for all $M \in \mathbb{N}$ so $\dim_H(D_m) = 1$ by Lemma 2.5. ☐

3.2 Proof of (ii)

Let $a \geq 0$, for any function $\varphi : [a, \infty) \to (0, \infty)$ satisfying (1.1) and (1.2), we denote by $f = \varphi^{-1}$ and specify the Moran set $A(\{\ell_n, m_n\}_{n=1}^{\infty})$ in Lemma 2.4 by setting

$$\ell_n = 2\left[1 + \frac{f(n) - f(n-1)}{2}\right] \quad \text{and} \quad m_n = 1 \tag{3.1}$$

with the convention that $f(0) = 0$, then it follows that

$$f(n) \leq \sum_{k=1}^{n} \ell_k \leq f(n) + 2n.$$

If $x \in A(\{\ell_n, m_n\}_{n=1}^{\infty})$, and $\sum_{k=1}^{n} \ell_k \leq N < \sum_{k=1}^{n+1} \ell_k$, then

$$n \leq \varphi(N) \leq \varphi(f(n+1) + 2(n+1)) = n + 1 + 2(n+1)o(1).$$

In the above, the last equality comes from the mean value theorem and the conditions

$$\lim_{t\to\infty} f(t) = +\infty \quad \text{and} \quad \lim_{t\to\infty} \varphi'(t) = 0.$$

By (3.1) and (a) of Lemma 2.4, one has that $n \leq R_N(x) \leq n + 1$, therefore

$$\lim_{N\to\infty} \frac{R_N(x)}{\varphi(N)} = 1,$$

hence,

$$A(\{\ell_n, m_n\}_{n=1}^{\infty}) \subset E_{\varphi} = \left\{x \in \{\pm 1\}^{\mathbb{N}} : \lim_{n\to\infty} \frac{R_n(x)}{\varphi(n)} = 1\right\}.$$

Now it remains to estimate the Hausdorff dimension of $A(\{\ell_n, m_n\}_{n=1}^{\infty})$ from below.
By (1.2) and (3.1), one has

$$\lim_{n\to\infty} \frac{\log \ell_{n+1}}{\ell_1 \cdots + \ell_n} \leq \lim_{n\to\infty} \frac{\log f(n+1)}{f(n)} = 0.$$

Now one can apply (c) of Lemma 2.4 and conclude that

$$\dim_H(A(\{\ell_n, m_n\}_{n=1}^{\infty})) = 1$$

as desired. □

3.3 Proof of (iii)

We treat two cases separately.

Case 1. $\gamma = 0$. Since $D_m \subset F_0$, by (i) one has that

$$\dim_H(F_0) \geq \dim_H(D_m) = 1 = \frac{h(0)}{\log 2}$$

implying the desired result.

Case 2. $0 < \gamma \leq \frac{1}{2}$. We first provide the upper bound by a covering argument, and then construct suitable Moran sets to give the lower bound.

The upper bound. For any $0 < \gamma \leq \frac{1}{2}$ and $0 < \varepsilon < \gamma$, put

$$\mathcal{F}_n := \left\{(x_1, \ldots, x_n) : \left|\frac{R_n(x)}{n} - \gamma\right| < \varepsilon\right\}. \tag{3.2}$$

Using the notation in (2.2), we write

$$F_n := [\mathcal{F}_n] = \bigcup_{\omega\in\mathcal{F}_n} [\omega] = \bigcup_{(\gamma-\varepsilon)n<m<(\gamma+\varepsilon)n} [\mathcal{C}_n(m)].$$

By definition, if $x \in F_\gamma$, then

$$\lim_{n\to\infty} \frac{R_n(x)}{n} = \gamma.$$

It follows that

$$F_\gamma \subset \bigcup_{J=1}^{\infty} \bigcap_{j=J}^{\infty} F_j,$$

which, in turn, implies

$$\dim_P(F_\gamma) \le \sup_{J\ge 1} \left\{ \dim_P \left(\bigcap_{j=J}^{\infty} F_j \right) \right\} \le \sup_{J\ge 1} \left\{ \overline{\dim}_B \left(\bigcap_{j=J}^{\infty} F_j \right) \right\}.$$

Notice that for each $J \ge 1$, F_n is a covering of $\bigcap_{j=J}^{\infty} F_j$ with n-cylinders (of diameter 2^{-n}) whenever $n \ge J$. Since the number of these cylinders, is just the cardinality of $\mathcal{F}_n$, one has

$$\dim_P(F_\gamma) \le \limsup_{n\to\infty} \frac{\log \sharp\mathcal{F}_n}{n \log 2}. \tag{3.3}$$

From (3.2), we see that

$$\sharp\mathcal{F}_n = \sum_{(\gamma-\varepsilon)n<m<(\gamma+\varepsilon)n} \sharp\mathcal{C}_n(m).$$

For each $(\gamma-\varepsilon)n < m < (\gamma+\varepsilon)n$, the assertion (a) of Lemma 2.2 gives

$$\log \sharp\mathcal{C}_n(m) \le nh(\gamma-\varepsilon) + o(n),$$

and the total number of such m's are no more than $2\varepsilon n$, therefore

$$\log \sharp\mathcal{F}_n \le nh(\gamma-\varepsilon) + o(n).$$

In combination with (3.3), this yields

$$\dim_P(F_\gamma) \le \frac{h(\gamma-\varepsilon)}{\log 2}.$$

This is true for arbitrarily small $\varepsilon > 0$ so

$$\dim_P(F_\gamma) \le \frac{h(\gamma)}{\log 2}.$$

The lower bound. We begin by specifying the Moran set $A(\{\ell_n, m_n\}_{n=1}^{\infty})$ in Lemma 2.4 with

$$\ell_n = 2n, \quad m_n = [2\gamma n].$$

Suppose that $x \in A(\{\ell_n, m_n\}_{n=1}^{\infty})$ and $\sum_{k=1}^{n} \ell_k \le N < \sum_{k=1}^{n+1} \ell_k$, then using (a) of Lemma 2.4, one gets

$$\sum_{k=1}^{n} m_k \le R_N(x) \le \sum_{k=1}^{n+1} m_k.$$

By the choices of ℓ_k and m_k, if $n(n+1) \leq N < (n+1)(n+2)$, then

$$\gamma n(n+1) - n \leq R_N(x) \leq \gamma(n+1)(n+2).$$

It follows that

$$\lim_{N\to\infty} \frac{R_N(x)}{N} = \gamma,$$

which, in turn, implies $A(\{\ell_n, m_n\}_{n=1}^{\infty}) \subset F_\gamma$.

Now it remains to estimate the Hausdorff dimension of $A(\{\ell_n, m_n\}_{n=1}^{\infty})$ from below. By (b) of Lemma 2.2, one has

$$\log \sharp\widetilde{\mathcal{C}}_{\ell_n}(m_n) = 2nh(\gamma) + o(n).$$

Putting this into (b) of Lemma 2.4, one gets

$$\dim_H(A(\{\ell_n, m_n\}_{n=1}^{\infty})) \geq \frac{h(\gamma)}{\log 2},$$

as desired. □

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